Exactly solvable three-level two-mode model with multiphoton transitions

1985 J. Phys. A: Math. Gen. 18 L1031

(http://iopscience.iop.org/0305-4470/18/16/007)

J. Phys. A: Math. Gen. 18 (1985) L1031-L1036. Printed in Great Britain

LETTER TO THE EDITOR

Exactly solvable three-level two-mode model with multiphoton transitions

A S Shumovsky†, E I Aliskenderov‡ and Fam Le Kien§

† Joint Institute for Nuclear Research, Dubna, USSR
‡ Institute of Cosmic Research, Baku, USSR
§ Moscow State University, Moscow, USSR

Received 22 August 1985

Abstract. The solutions of the equations of motion for the level population and photon number operators are obtained. The characteristic and photon distribution functions, the statistical moments of photon numbers and the correlations of modes are found.

The Jaynes-Cummings model [1] of a two-level atom interacting with a quantised single-mode radiation field is at the core of many problems in quantum optics, NMR and quantum electronics. The importance of this model lies in that it is perhaps the simplest solvable model which describes the essential physics of radiation-matter interaction. Recent studies of this model by Eberly *et al* [2] and Knight and Radmore [3] have revealed quantum collapse and revival which clearly are a manifestation of the role of quantum mechanics in the coherence and fluctuation properties of radiation-matter systems. In a series of papers Buck and Sukumar [4-7] and Singh [8] have proposed three exactly solvable generations of the Jaynes-Cummings model: one involving intensity dependent coupling, one involving multiphoton interaction between the field and atom, and the third involving few-level structure of the atom. A generalised model describing a two-mode process in a three-level atom with one-photon transitions has been investigated by Li and Bei [9] and Bogolubov *et al* [10, 11a].

The possibility of a multiphoton transition, which proceeds via intermediate states, has been first pointed out by Meyer [12]. Various multiphoton transition processes have been studied both theoretically and experimentally. Among them are two-photon and more general multiphoton lasers [13-19], two-photon decay [20, 21], multiphoton absorption and emission in a two-level atomic system [22, 23], Raman and hyper-Raman processes [24, 25].

We wish to present in this letter a rigorous and fully quantum mechanical treatment of multiphoton two-mode processes on a three-level atom on the basis of an exactly solvable Jaynes-Cummings-type model.

The three-level atomic model considered here is shown in figure 1 (for the case $m_1=2$, $m_2=1$). Let the upper level 3 be coupled with level 1 (level 2) due to the interaction with the field in mode 1 (mode 2) via m_1-photon (m_2-photon) transition. The model Hamiltonian of the system under consideration is

$$H = H_{\mathrm{A}} + H_{\mathrm{F}} + H_{\mathrm{AF}}. \tag{1}$$

Here H_{A} and H_{F} describe the free atom and free field respectively, and H_{AF} describes

0305-4470/85/161031+06$02.25

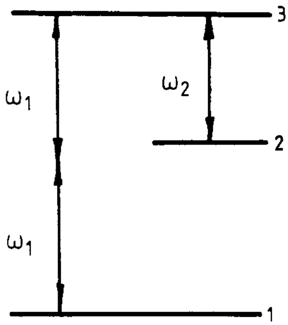

Figure 1.

the atom–field interaction in the dipole and rotating wave approximations

$$H_{\mathrm{A}} = \sum_{j=1}^{3} \hbar\Omega_j \hat{R}_{jj}, \qquad H_{\mathrm{F}} = \sum_{\alpha=1}^{2} \hbar\omega_\alpha \hat{a}_\alpha^+ \hat{a}_\alpha,$$
$$H_{\mathrm{AF}} = \sum_{\alpha=1}^{2} \hbar g_\alpha (\hat{R}_{3\alpha}\hat{a}_\alpha^{m_\alpha} + \hat{R}_{\alpha 3}\hat{a}_\alpha^{+m_\alpha}). \tag{2}$$

The operator $\hat{R}_{jj} \equiv |j\rangle\langle j|$ describes the population of level j. The operator $\hat{R}_{ij} \equiv |i\rangle\langle j|$ describes the atomic transition from level j to level i $(i \neq j)$. The operators $\hat{R}_{ij} = |i\rangle\langle j|$, $(i, j = 1, 2, 3)$ obey the relations

$$\hat{R}_{ij}\hat{R}_{kl} = \hat{R}_{il}\delta_{kj}, \qquad [\hat{R}_{ij}, \hat{R}_{kl}] = \hat{R}_{il}\delta_{kj} - \hat{R}_{kj}\delta_{il},$$
$$\sum_{i=1}^{3} \hat{R}_{ii} = 1. \tag{3}$$

The photon operators $\hat{a}_\alpha$, $\hat{a}_\alpha^+$ describe two modes of the radiation field with the resonance frequencies

$$m_\alpha \omega_\alpha = \Omega_3 - \Omega_\alpha, \tag{4}$$

and g_α are the corresponding atom-mode couplings. Note the case $m_1 = m_2 = 1$ has been considered by Bogolubov *et al* [10, 11a]. In the special case when the second mode is excluded from consideration, i.e. when $g_2 = 0$, we can obtain from the Hamiltonian (1) the one examined by Buck and Sukumar [5, 7] and Singh [8].

Starting from the Hamiltonian (1) we write down the Heisenberg equations for various operators in the usual way, i.e. $\dot{\hat{\theta}} = (\mathrm{i}/\hbar)[H, \hat{\theta}]$. First of all we define for convenience the subsidiary operators

$$A_\alpha \equiv \mathrm{i}(\hat{R}_{3\alpha}\hat{a}_\alpha^{m_\alpha} - \hat{R}_{\alpha 3}\hat{a}_\alpha^{+m_\alpha}). \tag{5}$$

Then, the Heisenberg equations for the level-population operators $\hat{R}_{\alpha\alpha}$ and the photon-number operators $\hat{N}_\alpha = \hat{a}_\alpha^+ \hat{a}_\alpha$ $(\alpha = 1, 2)$ are quickly established

$$\dot{\hat{R}}_{\alpha\alpha}(t) = g_\alpha A_\alpha(t), \qquad \dot{\hat{N}}_\alpha(t) = m_\alpha g_\alpha A_\alpha(t). \tag{6a, b}$$

From these equations it follows that

$$\hat{N}_\alpha(t) - m_\alpha \hat{R}_{\alpha\alpha}(t) = \text{constant} \equiv \hat{M}_\alpha, \tag{7}$$

where $\hat{M}_\alpha$'s are constants of motion.

By using the relations (3) the Heisenberg equations for $\hat{A}_\alpha$ are found to be

$$g_\alpha \dot{\hat{A}}_\alpha(t) = 2g_\alpha^2 \frac{(\hat{M}_\alpha + m_\alpha)!}{\hat{M}_\alpha!}[1 - \hat{R}_{11}(t) - \hat{R}_{22}(t) - \hat{R}_{\alpha\alpha}(t)] - g_1 g_2 \hat{B}(t), \tag{8}$$

where

$$\hat{B} \equiv \hat{R}_{21}\hat{a}_1^{m_1}\hat{a}_2^{+m_2} + \hat{R}_{12}\hat{a}_1^{+m_1}\hat{a}_2^{m_2}. \tag{9}$$

The operator $\hat{B}$ obeys the equation of motion

$$\dot{\hat{B}}(t) = g_1 \frac{(\hat{M}_1 + m_1)!}{\hat{M}_1!}\hat{A}_2(t) + g_2 \frac{(\hat{M}_2 + m_2)!}{\hat{M}_2!}\hat{A}_1(t). \tag{10}$$

The equations (6a), (8) and (10) form a closed system of linear equations which has the following integral of motion:

$$g_1 g_2 \hat{B}(t) - \hat{\lambda}_1^2 \hat{R}_{22}(t) - \hat{\lambda}_2^2 \hat{R}_{11}(t) = \text{constant} \equiv \hat{K}. \tag{11}$$

Here the notation

$$\hat{\lambda}_\alpha^2 \equiv g_\alpha^2 \frac{(\hat{M}_\alpha + m_\alpha)!}{\hat{M}_\alpha!}, \tag{12}$$

has been introduced.

Let us now differentiate each of equations (6a) with respect to time. Taking into account (8) and the constant of motion (11) we get then

$$\begin{aligned} \ddot{\hat{R}}_{11}(t) + (4\hat{\lambda}_1^2 + \hat{\lambda}_2^2)\hat{R}_{11}(t) + 3\hat{\lambda}_1^2 \hat{R}_{22}(t) &= 2\hat{\lambda}_1^2 - \hat{K} \\ \ddot{\hat{R}}_{22}(t) + (4\hat{\lambda}_2^2 + \hat{\lambda}_1^2)\hat{R}_{22}(t) + 3\hat{\lambda}_2^2 \hat{R}_{11}(t) &= 2\hat{\lambda}_2^2 - \hat{K}. \end{aligned} \tag{13}$$

One can consider these second-order differential equations as a system of equations for bounded quantum oscillators generating nonlinear nutations of level populations and photon numbers [26] in our model.

The solutions of the system (13) can be easily represented in the form [10]

$$\begin{aligned} \hat{R}_{11}(t) &= \hat{\mu}(\cos\hat{\lambda}t - 1) + \hat{\beta}\sin\hat{\lambda}t + \hat{\lambda}_1^2[\hat{u}(\cos 2\hat{\lambda}t - 1) + \hat{v}\sin 2\hat{\lambda}t] + \hat{R}_{11}(0) \\ \hat{R}_{22}(t) &= -\hat{\mu}(\cos\hat{\lambda}t - 1) - \hat{\beta}\sin\hat{\lambda}t + \hat{\lambda}_2^2[\hat{u}(\cos 2\hat{\lambda}t - 1) + \hat{v}\sin 2\hat{\lambda}t] + \hat{R}_{22}(0) \end{aligned} \tag{14}$$

where the operator

$$\hat{\lambda} \equiv (\hat{\lambda}_1^2 + \hat{\lambda}_2^2)^{1/2}, \tag{15}$$

describes the nutation frequencies. The 'amplitude operators' $\hat{\mu}$, $\hat{\beta}$, $\hat{u}$, $\hat{v}$ are defined by the initial conditions as follows

$$\begin{aligned} \hat{\mu} &= [\hat{\lambda}^2(\hat{\lambda}_2^2\hat{R}_{11}(0) - \hat{\lambda}_1^2\hat{R}_{22}(0)) + (\hat{\lambda}_2^2 - \hat{\lambda}_1^2)\hat{K}]/\hat{\lambda}^4 \\ \hat{u} &= [\hat{\lambda}^2(1 - 2\hat{R}_{33}(0)) + \hat{K}]/2\hat{\lambda}^4 \\ \hat{\beta} &= (\hat{\lambda}_2^2 g_1 \hat{A}_1(0) - \hat{\lambda}_1^2 g_2 \hat{A}_2(0))/\hat{\lambda}^3 \\ \hat{v} &= (g_1\hat{A}_1(0) + g_2\hat{A}_2(0))/2\hat{\lambda}^3. \end{aligned} \tag{16}$$

By using the conservation laws (3) and (7) together with (14) we can obtain

$$\begin{aligned} \hat{R}_{33}(t) &= -\hat{\lambda}^2[\hat{u}(\cos 2\hat{\lambda}t - 1) + \hat{v}\sin 2\hat{\lambda}t] + \hat{R}_{33}(0), \\ \hat{N}_1(t) &= m_1\{\hat{\mu}(\cos\hat{\lambda}t - 1) + \hat{\beta}\sin\hat{\lambda}t + \hat{\lambda}_1^2[\hat{u}(\cos 2\hat{\lambda}t - 1) + \hat{v}\sin 2\hat{\lambda}t]\} + \hat{N}_1(0), \\ \hat{N}_2(t) &= m_2\{-\hat{\mu}(\cos\hat{\lambda}t - 1) - \hat{\beta}\sin\hat{\lambda}t + \hat{\lambda}_2^2[\hat{u}(\cos 2\hat{\lambda}t - 1) + \hat{v}\sin 2\hat{\lambda}t]\} + \hat{N}_2(0). \end{aligned} \tag{17}$$

Thus, we have found the solutions of the equations of motion for the level-population and photon-number operators in the Heisenberg picture. Because of that the operators $\hat{M}_\alpha$ and hence the operators $\hat{\lambda}_\alpha$ and $\hat{\lambda}$ are diagonal in the space of the basis states, we can use the solutions (14) and (17) as conventional means of finding the time dependence of the level populations and photon numbers. By using these solutions we can easily find also the statistical characteristics of the photons in the system (see [11b] and below).

Let us introduce the following operator of the characteristic function of photon distribution

$$\hat{\chi}(\xi_1, \xi_2) = \exp(\mathrm{i}\xi_1 \hat{N}_1(t) + \mathrm{o}\xi_2 \hat{N}_2(t)). \tag{18}$$

Using the conservation laws (7) we find

$$\hat{\chi}(\xi_1, \xi_2) = \exp(\mathrm{i}\xi_1 \hat{M}_1 + \mathrm{i}\xi_2 \hat{M}_2)\{[\exp(\mathrm{i}\xi_i m_1) - 1]\hat{R}_{11}(t) + [\exp(\mathrm{i}\xi_2 m_2) - 1]\hat{R}_{22}(t) + 1\}. \tag{19}$$

Denote by $\hat{\rho}(0)$ the density operator describing an initial state of the 'atom-field' system. Then the characteristic function $\hat{\chi}(\xi_1, \xi_2)$ is defined as

$$\langle \hat{\chi}(\xi_1, \xi_2)\rangle = \mathrm{Tr}\, \hat{\chi}(\xi_1, \xi_2)\hat{\rho}(0). \tag{20}$$

It is related with the photon distribution function $P(n_1, n_2; t)$ by

$$\chi(\xi_1, \xi_2) = \sum_{n_1 n_2} \exp(\mathrm{i}\xi_1 n_1 + \mathrm{i}\xi_2 n_2) P(n_1, n_2; t), \tag{21}$$

which allows us to obtain the latter if the former is known.

Once the characteristic and photon distribution functions are known, it is easy to find the statistical moments of photon number $\langle \hat{N}_\alpha^m(t)\rangle$ and the correlation of modes $\langle \hat{N}_1^k(t), \hat{N}_2^l(t)\rangle$ using the relations

$$\begin{aligned} \langle \hat{N}_\alpha^m(t)\rangle &= \sum_{n_1 n_2} n_\alpha^m P(n_1, n_2; t) = \frac{\partial^m}{\partial(i\xi_\alpha)^m}\langle \hat{\chi}(\xi_1 = 0, \xi_2 = 0)\rangle, \\ \langle \hat{N}_1^k(t), \hat{N}_2^l(t)\rangle &= \sum_{n_1 n_2} n_1^k n_2^l P(n_1, n_2; t) = \frac{\partial^{k+l}}{\partial(i\xi_1)^k \partial(i\xi_2)^l}\langle \hat{\chi}(\xi_1 = 0, \xi_2 = 0)\rangle. \end{aligned} \tag{22}$$

Equations (19)-(22) together with (14) allow us to discuss photon statistics for a given initial state of the system. A detailed consideration of this problem will be given below.

We first assume that the atom is initially on a level i, i.e.

$$\hat{\rho}(0) = |i\rangle\langle i| \otimes \hat{\rho}_{\mathrm{F}}, \tag{23}$$

where the density matrix $\hat{\rho}_{\mathrm{F}}$ describes the initial state of the field. Then, by using (19), (14) and (23) the characteristic function (20) is found to be

$$\langle \hat{\chi}(\xi_1, \xi_2)\rangle = \sum_{n_1 n_2} P(n_1, n_2) \exp[\mathrm{i}\xi_1(n_1 - m_1\delta_{1i}) + \mathrm{i}\xi_2(n_2 - m_2\delta_{2i})] \times \{[\exp(\mathrm{i}\xi_1 m_1) - 1]R_1(i, n_1, n_2; t) + [\exp(\mathrm{i}\xi_2 m_2) - 1]R_2(i, n_1, n_2; t) + 1\}. \tag{24}$$

Here $P(n_1, n_2)$ is the initial distribution of photon numbers

$$P(n_1, n_2) = \langle n_2, n_1|\hat{\rho}_{\mathrm{F}}|n_1, n_2\rangle. \tag{25}$$

The functions $R_\alpha(i, n_1, n_2; t)$ in (24) are determined as

$$\begin{aligned} R_1(i, n_1, n_2; t) &= -2\mu(i, n_1, n_2)\sin^2\tfrac{1}{2}\lambda(i, n_1, n_2)t \\ &\quad -2\lambda_1^2(i, n_1, n_2)u(i, n_1, n_2)\sin^2\lambda(i, n_1, n_2)t + \delta_{1i}, \\ R_2(i, n_1, n_2; t) &= 2\mu(i, n_1, n_2)\sin^2\tfrac{1}{2}\lambda(i, n_1, n_2)t \\ &\quad -2\lambda_2^2(i, n_1, n_2)u(i, n_1, n_2)\sin^2\lambda(i, n_1, n_2)t + \delta_{2i}, \end{aligned} \tag{26}$$

where

$$\begin{aligned} \lambda_\alpha(i, n_1, n_2) &= g_\alpha\left(\frac{(n_\alpha - m_\alpha\delta_{\alpha i} + m_\alpha)!}{(n_\alpha - m_\alpha\delta_{\alpha i})!}\right)^{1/2}, \\ \lambda(i, n_1, n_2) &= (\lambda_1^2(i, n_1, n_2) + \lambda_2^2(i, n_1, n_2))^{1/2}, \\ \mu(i, n_1, n_2) &= 2\lambda_1^2(i, n_1, n_2)\lambda_2^2(i, n_1, n_2)(\delta_{1i} - \delta_{2i})/\lambda^4(i, n_1, n_2), \\ u(i, n_1, n_2) &= (\lambda_1^2(i, n_1, n_2)\delta_{1i} + \lambda_2^2(i, n_1, n_2)\delta_{2i} \\ &\quad -\lambda^2(i, n_1, n_2)\delta_{3i})/2\lambda^4(i, n_1, n_2). \end{aligned} \tag{27}$$

Comparing (24) with (21) we obtain

$$\begin{aligned} P(n_1, n_2; t) &= P(n_1 + m_1\delta_{1i} - m_1, n_2 + m_2\delta_{2i}) \\ &\quad \times R_1(i, n_1 + m_1\delta_{1i} - m_1, n_2 + m_2\delta_{2i}; t) + P(n_1 + m_1\delta_{1i}, n_2 + m_2\delta_{2i} - m_2) \\ &\quad \times R_2(i, n_1 + m_1\delta_{1i}, n_2 + m_2\delta_{2i} - m_2; t) + P(n_1 + m_1\delta_{1i}, n_2 + m_2\delta_{2i}) \\ &\quad \times R_3(i, n_1 + m_1\delta_{1i}, n_2 + m_2\delta_{2i}; t), \end{aligned} \tag{28}$$

where

$$R_3(i, n_1, n_2; t) = 2\lambda^2(i, n_1, n_2)u(i, n_1, n_2)\sin^2\lambda(i, n_1, n_2)t + \delta_{3i}. \tag{29}$$

The statistical moments of photon numbers and the correlations of modes are found from (22) and (24) to be

$$\begin{aligned} \langle \hat{N}_\alpha^m(t)\rangle &= \sum_{n_1 n_2} P(n_1, n_2)\{(n_\alpha - m_\alpha\delta_{\alpha i})^m \\ &\quad + [(n_\alpha - m_\alpha\delta_{\alpha i} + m_\alpha)^m - (n_\alpha - m_\alpha\delta_{\alpha i})^m]R_\alpha(i, n_1, n_2; t)\}, \end{aligned} \tag{30}$$

$$\begin{aligned} \langle \hat{N}_1^k(t), \hat{N}_1^l(t)\rangle &= \sum_{n_1 m_1} P(n_1, n_2)\{(n_1 - m_1\delta_{1i})^k(n_2 - m_2\delta_{2i})^l \\ &\quad + (n_1 - m_1\delta_{1i})^k[(n_2 - m_2\delta_{2i} + m_2)^l - (n_2 - m_2\delta_{2i})^l]R_2(i, n_1, n_2; t) \\ &\quad + (n_2 - m_2\delta_{2i})^l[(n_1 - m_1\delta_{1i} + m_1)^k - (n_1 - m_1\delta_{1i})^k]R_1(i, n_1, n_2; t). \end{aligned}$$

In particular, we find

$$\langle \hat{N}_\alpha(t)\rangle = \sum_{n_1 n_2} P(n_1, n_2)(n_\alpha - m_\alpha\delta_{\alpha i} + m_\alpha R_\alpha(i, n_1, n_2; t)), \tag{31}$$

$$\langle \hat{N}_\alpha^2(t)\rangle = \sum_{n_1 n_2} P(n_1, n_2)\{(n_\alpha - m_\alpha\delta_{\alpha i})^2 + [2m_\alpha(n_\alpha - m_\alpha\delta_{\alpha i}) + m_\alpha^2]R_\alpha(i, n_1, n_2; t)\},$$

$$\begin{aligned} \langle \hat{N}_1(t), \hat{N}_2(t)\rangle &= \sum_{n_1 n_2} P(n_1, n_2)\{(n_1 - m_1\delta_{1i})(n_2 - m_2\delta_{2i}) \\ &\quad + (n_1 - m_1\delta_{1i})m_2R_2(i, n_1, n_2; t) + (n_2 - m_2\delta_{2i})m_1R_1(i, n_1, n_2; t)\}. \end{aligned}$$

Note in the case $i=1$, $m_1=m_2=1$ equations (31) reduce to the results obtained by Bogolubov *et al* [11b]. Equation (28) for the distribution function of photon numbers can easily be found in other ways using either the time evolution operators in the Schrödinger picture or the dressed state formalism for calculating the transition probabilities of the atom.

Thus, in this letter we have obtained the exact solution of equations of motion for the level population and photon number operators. The characteristic and photon distribution functions, the statistical moments of photon numbers and the correlations of modes have been found.

References

[1] Jaynes E T and Cummings F W 1963 *Proc. IEEE* **51** 89
[2] Eberly J H, Narozhny N B and Sancher-Mondragon J J 1981 *Phys. Rev.* A **23** 236
[3] Knight P L and Radmore P M 1982 *Phys. Lett.* **90A** 342
[4] Buck B and Sukumar C V 1981 *Phys. Lett.* **81A** 132
[5] Sukumar C V and Buck B 1981 *Phys. Lett.* **83A** 211
[6] Buck B and Sukumar C V 1984 *J. Phys. A: Math. Gen.* **17** 877
[7] Sukumar C V and Buck B 1984 *J. Phys. A: Math. Gen.* **17** 885
[8] Singh S 1982 *Phys. Rev.* A **25** 3206
[9] Li X and Bei N 1984 *Phys. Lett.* **101A** 169
[10] Bogolubov N N Jr, Fam Le Kien and Shumovsky A S 1984 *Phys. Lett.* **101A** 201
[11a] Bogolubov N N Jr, Fam Le Kien and Shumovsky A S 1985 *Phys. Lett.* **107A** 173
[11b] Bogolubov N N Jr 1985 *Phys. Lett.* **107A** 456
[12] Goeppert-Meyer M 1931 *Ann. Phys., Lpz.* **9** 273
[13] McNeil K J and Walls D F 1975 *J. Phys. A: Math. Gen.* **8** 104
[14] Sczaniecky L 1980 *Opt. Acta* **27** 251
[15] Gibson A F and Key M H 1980 *Rep. Prog. Phys.* **43** 3
[16] Sharma M P and Brescansin L M 1981 *Phys. Rev.* A **23** 1893
[17] Reid M, McNeil K J and Walls D F 1981 *Phys. Rev.* A **24** 2029
[18] Zubairy M S 1982 *Phys. Lett.* **87A** 162
[19] Wang Z C and Haken H 1984 *Z. Phys.* B **55** 361; 1984 *Z. Phys.* B **56** 77
[20] Tung J H, Ye X M, Salamo G J and Chan F T 1984 *Phys. Rev.* A **30** 1175
[21] Florescu V 1984 *Phys. Rev.* A **30** 2441
[22] Shen Y R 1967 *Phys. Rev.* **155** 921
[23] Zubairy M S and Yeh J J 1980 *Phys. Rev.* A **21** 1624
[24] Simaan H D 1978 *J. Phys. A: Math. Gen.* **11** 1799
[25] Sainz de los Terreros L, Santos M and Gonzabz-Diaz P F 1985 *Phys. Rev.* A **31** 1598
[26] Elgin J N 1980 *Phys. Lett.* **80A** 140